

*"If I have seen further it is by standing
on the shoulders of giants."*

Isaac Newton, 1675

3 Two-Fluid Model

Simple isothermal two-fluid two-phase models for stratified flow:

- ➔ Mass and momentum conservation
 - ➔ Friction and pressure loss
 - ➔ Simplifications and solution for steady-state incompressible flow
 - ➔ Simplifications and solution for steady-state compressible flow
 - ➔ Fully transient solution
 - ➔ Simplifying by introducing the drift-flux model
 - ➔ Further simplification by ignoring inertia in the drift-flux model
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3.1 Problem definition

Let us now look at an example of how to utilize the very general multi-phase model from chapter 2. To avoid getting lost in details, we study a relatively simple situation:

- We have only two fluids, one gas and one liquid, and pressures and temperatures are such that evaporation or condensation does not occur. We also assume no gas can be dissolved in the liquid (even though this is never quite true, as liquids do take up some gas in the same way oxygen is taken up by water, enabling fish to breathe).
- The pipe has no perforations, so neither liquid nor gas can flow through the pipe wall.
- The flow is stratified – we simply neglect all other flow regimes for now.

- The flow is isothermal, so we do not need the energy equation to keep track of the temperature.

With these simplifications, let us try to establish all necessary conservation equations. Also, we will develop closure relationships, which in this highly simplified case are reduced to describing the frictions between the gas and the pipe wall, between the liquid and the pipe wall, and between the gas and the liquid, in addition to some fluid properties. To make the equation system hyperbolic, we also need to describe the pressure difference between the gas and the liquid.

3.2 Mass conservation

If we use index G for gas and L for liquid, we can write two mass conservation equations based on equation 2.2.4. For the gas phase, we get:

$$\frac{\partial(\alpha_G \rho_G)}{\partial t} = -\frac{\partial(\alpha_G \rho_G v_G)}{\partial x} + \Gamma_{LG} + \Gamma_{GW} \quad (3.2.1)$$

Since we have no phase change, $\Gamma_{LG} = 0$. Also, no gas is going to be added through the wall, and we therefore set $\Gamma_{GW} = 0$. Equation 3.2.1 simplifies to:

$$\frac{\partial(\alpha_G \rho_G)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G)}{\partial x} = 0 \quad (3.2.2)$$

Similarly, mass conservation for the liquid becomes:

$$\frac{\partial(\alpha_L \rho_L)}{\partial t} + \frac{\partial(\alpha_L \rho_L v_L)}{\partial x} = 0 \quad (3.2.3)$$

Equation 2.2.6 is very simple in this case:

$$\alpha_G + \alpha_L = 1 \quad (3.2.4)$$

3.3 Momentum conservation

Equation 2.3.17 applied to the gas-phase leads to:

$$\begin{aligned} \frac{\partial(\alpha_G \rho_G v_G)}{\partial t} = & -\frac{\partial(\alpha_G \rho_G v_G^2)}{\partial x} - \alpha_G \frac{\partial p_G}{\partial x} + R_{LG} + R_{GW} \\ & + S_{LG} + S_{GW} - \alpha_G \rho_G g \sin \theta \end{aligned} \quad (3.3.1)$$

R_{LG} is friction force pr. unit pipe volume from liquid on the gas, and R_{GW} is similarly volume-specific friction force from the wall on the gas.

Assuming all surface tension forces acting directly on the gas flow are negligible, a good approximation for stratified flow, we can set $S_{LG} = S_{GW} = 0$. In addition, we define the pressure on the interface (the liquid surface) between the gas and liquid as p , while Δp_G is the extra pressure felt by the gas due to its average elevation being different from that of the interface (Δp_G is obviously going to be negative, given that the gas is on top of the interface).

$$\frac{\partial(\alpha_G \rho_G v_G)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G^2)}{\partial x} = -\alpha_G \frac{\partial(p + \Delta p_G)}{\partial x} - R_{GL} + R_{GW} - \alpha_G \rho_G g \sin \theta \quad (3.3.2)$$

For the liquid, we similarly get:

$$\frac{\partial(\alpha_L \rho_L v_L)}{\partial t} + \frac{\partial(\alpha_L \rho_L v_L^2)}{\partial x} = -\alpha_L \frac{\partial(p + \Delta p_L)}{\partial x} + R_{GL} + R_{LW} - \alpha_L \rho_L g \sin \theta \quad (3.3.3)$$

We notice that equations 3.3.2 and 3.3.3 also satisfy the requirement that the sum of all forces between phases must be zero (equation 2.3.17), since R_{GL} occurs with opposite sign in equation 3.3.2 and 3.3.3.

3.4 Gas and liquid pressure difference in stratified flow

In this simple model, it would be tempting to neglect the pressure correction terms (setting $\Delta p_G = \Delta p_L = 0$), meaning all pressures in a cross-section would be equal so $p_G = p_L = p$. As explained in chapter 2.3.2, however, this would neglect the mechanisms creating surface waves while creating an un-physical system description which in turn can cause loss of hyperbolicity and numerical problems.

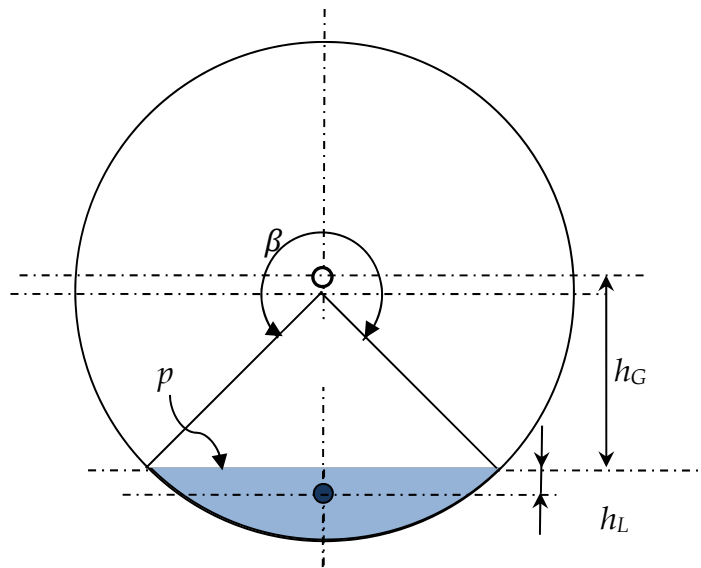


Figure 3.4.1. Stratified flow. Center of gravity for the gas is above the pipe center, while for the liquid, it is of course somewhere below the interface surface.

A pipe's circular cross-section leads to somewhat different wave conditions compared to the surface of a lake. If a wave-top rises above the pipe's center line, it does so in a diminishing cross-section. Therefore it closes the remaining cross-section relatively fast

when approaching the upper side of the pipe. That affects the wave pattern significantly, and it is clearly worthwhile implementing a relatively accurate description of the circular geometry to get this right.

For stratified flow we assume the interface in each cross-section to be a straight, horizontal line. That is a good approximation for low gas velocities, but measurements have shown that increasing velocities make the surface bend until the liquid covers the whole wall for fully annular flow.

One can calculate the pressure at a point h_L below the interface in alternative ways. Some writers propose that the pressure difference is a function only of the difference in static head between interface and liquid area center of gravity (Taitel & Duckler 1976, Watson 1990, Barnea & Taitel 1993, 1996), while others propose taking into account the Bernoulli-effect resulting from the fact that different phases have different average velocities (Tuomi 1996, Coquel et al. 1997, Bestion 1990). The former approach seems to be the most correct, since pressure by definition must be the same in all directions – every point in space and time must necessarily comply with this, including points at the interface (at least in our case, where we have decided to neglect surface tension). The conservation equations describe the correlation between pressure and velocity, so the Bernoulli-effect is already built into them as they stand.

Each phase is modeled separately in stratified flow - they are only connected via friction, total cross-sectional area, and pressure. Therefore the pressure correction terms' mission is not to describe a (non-existent) pressure difference at the surface between the two, but the pressure difference between the two phases (at some average point for each phase). If the phases are distributed as shown in figure 3.4.1, the task comes down to describing the average elevation difference and resulting static pressure head between the phases. It is not self evident exactly what should be taken as average elevation for each phase, since the velocities vary across the cross-section (it is generally lower near the wall than it is elsewhere). For simplicity we use the *area center of gravity* for each phase as our elevation point.

It is worth noting that different authors use different reference levels as a basis for their pressure modification terms, some use the pipe bottom pressure as reference instead of the interface. That is not expected to affect the results, but some of the terms in the equations do of course appear to be somewhat different.

At first glance we would expect the required geometrical correlations to describe areas and elevations for a circular cross-section to be very simple, but on closer inspection they turn out to be relatively complicated. De Henau et al. (1995) have shown that:

$$h_G = \left[\frac{1}{2} \cos \left(\pi - \frac{\beta}{2} \right) + \frac{1}{3\pi\alpha_G} \sin^3 \left(\pi - \frac{\beta}{2} \right) \right] d \cos \theta \quad (3.4.1)$$

$$h_L = \left[-\frac{1}{2} \cos \left(\pi - \frac{\beta}{2} \right) + \frac{1}{3\pi\alpha_L} \sin^3 \left(\pi - \frac{\beta}{2} \right) \right] d \cos \theta \quad (3.4.2)$$

The pressure differences can then easily be calculated as:

$$\Delta p_G = p_G - p = -\rho_G g h_G \quad (3.4.3)$$

$$\Delta p_L = p_L - p = \rho_L g h_L \quad (3.4.4)$$

The angle β , as defined on figure 3.4.1, must be estimated from how full the pipe is, meaning from α_G and α_L . That angle is also useful when determining the various surfaces involved in the friction calculations. An accurate explicit description of the function (α_G, α_L) is not known, but it is possible to express an equation which can be solved to any required accuracy using Newton-iteration. Given the inaccuracies introduced on various points when developing this model, a more direct but not completely accurate approximation proposed by Biberg (1999) should suffice, it is claimed to be accurate to within $\pm 0.002 \text{ rad}$:

$$\beta = 2\pi - 2 \left\{ \pi\alpha_L + \left(\frac{3\pi}{2} \right)^{1/3} \left[1 - 2\alpha_L + \alpha_L^{1/3} - (1 - \alpha_L)^{1/3} \right] \right\} \quad (3.4.5)$$

When $\theta = \pi/2$ (vertical pipe), $\Delta p_G = \Delta p_L = 0$, which can lead to loss of hyperbolicity. We cannot expect this model to describe surface waves in nearly vertical pipes accurately anyway (such waves are affected by surface tension more than gravity), and

we may as well limit $\cos \theta$ in the pressure correction terms in equations 3.4.1 and 3.4.2 (though not elsewhere) so we use $\max[(\cos \theta)_{min}, \cos \theta]$ instead of $\cos \theta$. That way, $\cos \theta$ never falls below a certain value, for instance by setting $(\cos \theta)_{min} = 0.1$.

For our simulations to represent surface waves realistically, we have to use a very dense grid. Surface waves are quite short compared to long pipelines, and they obviously cannot be described by grid points which are further apart than half the wave length. It is not convenient to use such a dense grid in most flow assurance simulations, but, as we will discover in chapter 9, very fine grid simulations may become a practical way to describe surface waves and the onset of slugging in future models.

3.5 Friction in stratified flow

The friction between gas and pipe wall, R_{GW} , is difficult to express accurately. We remember from *Pipe Flow 1* chapter 2 and 3 that the Darcy-Weissbach friction factor can be relatively inaccurate even for circular pipes with single-phase flow. For the friction between gas and liquid, an additional difficulty comes from the fact that we do not know the surface roughness on the interface. Friction errors will also lead to incorrect volume fractions, which again affect the friction calculations. We must therefore expect estimates of the interface friction to be considerably less accurate than previous friction calculations for single-phase flow, where we encountered errors as high as 20%. Keeping these limitations in mind, we will try to develop reasonably accurate estimates for the Darcy-Weissbach friction factors.

Hydraulic diameter can be defined from the wetted perimeter of the cross-section, O , and the cross-sectional area, A , as:

$$d_h \stackrel{\text{def}}{=} \frac{4A}{O} \quad (3.5.1)$$

We may also recall that for non-circular cross-sections, it is generally necessary to include a geometric correction factor in friction calculations for non-circular cross-sections, but we are going to presume that factor to be 1 in our simple model. That is probably not a bad approximation compared to the other simplifications we are using here.

By studying figure 3.4.1, we see that wetted perimeter relevant to the friction between gas and pipe wall is:

$$O_{GW} = \beta \frac{d}{2} \quad (3.5.2)$$

Between liquid and pipe wall:

$$O_{LW} = (2\pi - \beta) \frac{d}{2} \quad (3.5.3)$$

Interface between gas and liquid:

$$O_{GL} = d \sin\left(\frac{\beta}{2}\right) \quad (3.5.4)$$

Gas cross-sectional area:

$$A_G = \alpha_G \frac{\pi d^2}{4} \quad (3.5.5)$$

Liquid cross-sectional area:

$$A_L = \alpha_L \frac{\pi d^2}{4} \quad (3.5.6)$$

From this, we define the three relevant hydraulic diameters. For calculating the gas-wall Reynolds number, we use the following hydraulic diameter:

$$d_{h\ GW} \stackrel{\text{def}}{=} \frac{4A_G}{O_{GW}} \quad (3.5.7)$$

For calculating the liquid-wall hydraulic diameter:

$$d_{h\ LW} \stackrel{\text{def}}{=} \frac{4A_L}{O_{LW}} \quad (3.5.8)$$

The Reynolds numbers can then be defined accordingly as:

$$Re_{h\ GW} \stackrel{\text{def}}{=} \frac{v_G d_{h\ GW} \rho_G}{\mu_G} \quad (3.5.9)$$

$$Re_{h\ LW} \stackrel{\text{def}}{=} \frac{v_L d_{h\ LW} \rho_L}{\mu_L} \quad (3.5.10)$$

Which area to select when defining the hydraulic diameter for the interface between liquid and gas is less obvious, since both the gas and liquid area seem to be involved. Also, keep in mind that the interface friction factor in reality is not entirely independent of the wall friction factors, since each affect both turbulence and velocity profiles and thereby also the friction mechanism. The most used empirical correlation to estimate the interfacial friction factor is probably the one proposed by Petalas & Aziz (1997):

$$f_{GL} = (0.004 + 0.5 \cdot 10^{-6} Re_{SL}) Fr_L^{1.335} \frac{\rho_L d g}{\rho_G v_G^2} \quad (3.5.11)$$

Where Re_{SL} is the liquid phase Reynolds number based on superficial velocity ($= \alpha_L v_L$). The liquid's Froude number is defined by the liquid height h_L as:

$$Fr_L = \frac{v_L}{\sqrt{gh_L}} \quad (3.5.12)$$

The gas-wall friction shear τ_{Gw} becomes:

$$\tau_{Gw} = -\frac{f_{Gw} \rho_G}{8} v_G |v_G| \quad (3.5.13)$$

R_{Gw} is defined as friction force ($= \tau_{Gw} O_{Gw} \Delta x$) pr. unit volume ($= A_G \Delta x$). We get:

$$R_{GW} = -\frac{f_{GW}\rho_G}{8} \frac{O_{GW}}{A} v_G |v_G| \quad (3.5.14)$$

Similar for the liquid-wall friction :

$$R_{LW} = -\frac{f_{LW}\rho_L}{8} \frac{O_{LW}}{A} v_L |v_L| \quad (3.5.15)$$

If we estimate the relevant gas-liquid velocity difference as $v_G - v_L$, the interfacial friction is estimated as:

$$R_{GL} = \frac{f_{GL}\rho_G}{8} \frac{O_{GL}}{A} (v_G - v_L) |v_G - v_L| \quad (3.5.16)$$

We can then estimate all three frictions easily as follows:

1. Use equation 3.4.5 to calculate β .
2. Use equations 3.5.2 - 3.5.10 to define the Reynolds numbers $Re_{h_{GW}}$ and $Re_{h_{LW}}$.
3. Use the Modified Moody Diagram (or its numerical representation, as outlined in table 2.13.3 in *Pipe Flow 1*) to estimate the Darcy-Weisbach friction factors f_{GW} and f_{LW} . For the interface friction factor f_{GL} , use equations 3.5.12 and 3.5.11.
4. Calculate R_{GW} , R_{LW} and R_{GL} according to equations 3.5.14 - 3.5.16.

3.6 Steady-state incompressible flow solution

3.6.1 The model

As a first approach to solving the equations, let us introduce the following simplifications (in addition to the ones already listed in chapter 3.1):

1. The flow is steady-state. That means nothing changes over time, and so the time derivatives in the conservation equations 3.2.2, 3.2.3, 3.3.2, and 3.3.3 are all going to be zero.
2. All gas and liquid properties are independent of the pressure. This means both fluids are considered incompressible with constant viscosity. That is usually not a good approximation for the gas in pipelines and is only done for convenience at this step in the process of familiarizing ourselves with the equations.
3. The pipe's elevation angle θ is constant.

As boundary conditions we impose constant mass flows for both gas and liquid at the inlet and a constant pressure at the outlet. We can use constants $k_{G\ in}$ and $k_{L\ in}$ for defining the inlet boundary conditions:

$$\alpha_G \rho_G v_G = k_{G\ in} \quad (3.6.1)$$

$$\alpha_L \rho_L v_L = k_{L\ in} \quad (3.6.2)$$

Since the flow is steady-state with no phase change and no fluid flows in through perforations, the mass flow must be constant along the entire pipeline. Equations 3.6.1 and 3.6.2 are therefore not restricted to the inlet – they are valid everywhere.

When the mass conservation equation 3.2.2 is stripped of the time derivative, it becomes:

$$\frac{\partial(\alpha_G \rho_G v_G)}{\partial x} = 0 \quad (3.6.3)$$

This can be written as:

$$\alpha_G \rho_G \frac{\partial v_G}{\partial x} + \rho_G v_G \frac{\partial \alpha_G}{\partial x} + \alpha_G v_G \frac{\partial \rho_G}{\partial x} = 0 \quad (3.6.4)$$

Since the density is constant, the last term on the left-hand side is zero, and we end up with

$$\alpha_G \frac{\partial v_G}{\partial x} + v_G \frac{\partial \alpha_G}{\partial x} = 0 \quad (3.6.5)$$

This means any change in gas fraction must always be offset by a similar change in gas velocity to satisfy the mass balance. But since no property changes with pressure, the absolute pressure cannot affect the flow at any point in the pipe, only the pressure loss per unit length can. The same goes for all other parameters affecting the flow, such as the frictions and the elevations: They all affect each point along the pipe in a constant way, and therefore there is nothing which can favor an increase or decrease in α_G (and a corresponding variation in v_G) along the pipeline. This implies that α_G and v_G must be constant from inlet to outlet. Only the pressure changes along the pipe, and all spatial derivatives not containing the pressure vanish. The continuity equations only express $0 = 0$, and we are left with the momentum equations.

Since this very simplified model cannot describe surface waves on the liquid's surface anyway and has no hyperbolicity to maintain, we neglect the pressure difference between phases by setting $\Delta p_G = \Delta p_L = 0$, and the momentum equations 3.3.2 and 3.3.3 are reduced to:

$$0 = -\alpha_G \frac{\partial p}{\partial x} - R_{GL} + R_{GW} - \alpha_G \rho_G g \sin \theta \quad (3.6.6)$$

$$0 = -\alpha_L \frac{\partial p}{\partial x} + R_{GL} + R_{GW} - \alpha_L \rho_L g \sin \theta \quad (3.6.7)$$

We can use these two equations to eliminate $\partial p / \partial x$:

$$-\frac{R_{GL}}{\alpha_G} + \frac{R_{GW}}{\alpha_G} - \rho_G g \sin \theta = \frac{R_{GL}}{\alpha_L} + \frac{R_{LW}}{\alpha_L} - \rho_L g \sin \theta \quad (3.6.8)$$

If we look at equations 3.5.14 – 3.5.16 and the relatively simple correlations they rely on, equations 3.4.5 and 3.5.2 – 3.5.12, we see that they can be solved relatively easily by inserting all underlying equations into equation 3.6.8.

We define a matrix based on equations 3.6.1, 3.6.2, 3.6.8, and 3.2.4:

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \alpha_G \rho_G v_G - k_{G \text{ in}} \\ \alpha_L \rho_L v_L - k_{L \text{ in}} \\ \frac{R_{GLi}}{\alpha_G} - \frac{R_{GW}}{\alpha_G} + \rho_G g \sin \theta + \frac{R_{GLi}}{\alpha_L} + \frac{R_{LW}}{\alpha_L} - \rho_L g \sin \theta \\ \alpha_G + \alpha_L - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.6.9)$$

The input variables are:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \alpha_G \\ \alpha_L \\ v_G \\ v_L \end{bmatrix} \quad (3.6.10)$$

3.6.2 Solution method

Newton-iteration on equation 3.6.9 and 3.6.10 is straight forward:

$$Y_{n+1} = Y_n - J^{-1} F_n(Y_n) \quad (3.6.11)$$

We start by guessing a likely value for α_G , and we set $\alpha_L = 1 - \alpha_G$. This is used to determine starting values for v_G and v_L in such a way that they satisfy the inlet boundary conditions in equations 3.6.1 and 3.6.2. Then it becomes possible to calculate starting values for the frictions by setting $R_{GL} = R_{GL}(\alpha_G, \alpha_L, v_G, v_L)$, $R_{GW} = R_{GW}(\alpha_G, \alpha_L, v_G, v_L)$, and $R_{LW} = R_{LW}(\alpha_G, \alpha_L, v_G, v_L)$. These values are used to estimate the Y and F -vectors.

The Jacobi-matrix $J = \partial F / \partial Y$ can be estimated by varying each argument slightly and then investigating how that affects F . We start by calculating F and then give α_G a slightly larger value, say $\alpha_G + \Delta\alpha_G$, where $\Delta\alpha_G$ may be in the order of 10^{-7} . All other arguments remain as they are, and a new F -matrix is calculated, let us call it $F_{\Delta\alpha_G}$. This enables us to estimate the first row in the Jacobi-matrix:

$$J_{11} = \frac{\partial f_1}{\partial y_1} \approx \frac{f_{1\Delta\alpha_G} - f_1}{\Delta\alpha_G}, J_{21} = \frac{\partial f_2}{\partial y_1} \approx \frac{f_{2\Delta\alpha_G} - f_2}{\Delta\alpha_G}, \dots \quad (3.6.12)$$

The same process is repeated for each of the arguments so that the whole Jacobi-matrix is estimated, and equation 3.6.11 is used to iterate. Careful programming is necessary to make the algorithm robust. The usual potential problems due to division by zero or attempting to take roots of negative numbers can otherwise occur.

The iteration process is repeated until convergence, typically less than 10 times. Since (according to equation 3.6.9) the F -vector should end up having zero length, some norm of F can be used as convergence criterion, for instance the absolute norm:

$$\sqrt{\sum_{i=1}^4 f_i^2} \leq e_F \quad (3.6.13)$$

where e_F is the maximum accepted error.

After convergence is achieved, we may want to know the pipe's pressure profile. Since all velocities, fractions and fluid properties are constant, the pressure loss is going to be a linear function of x , and we can accurately set:

$$\frac{\partial p}{\partial x} = \frac{\Delta p}{\Delta x} \quad (3.6.14)$$

This can be used to calculate the pressure difference Δp compared to the outlet pressure (or the inlet pressure, if that had been one of the known boundary conditions) at any distance Δx from the outlet by inserting equation 3.6.14 into 3.6.6 or 3.6.7.

3.7 Steady-state compressible flow solution

Our model becomes more useful if we abandon the requirement from chapter 3.6 that both fluids have to be incompressible. The spatial derivatives can no longer be neglected, but the flow is still steady-state, and all time derivatives continue to be zero. We see that the mass conservation equations 3.2.2 and 3.2.3 stripped of the time

derivative terms simply state that the mass flows are the same everywhere, which means that the boundary equations 3.6.1 and 3.6.2 still will be valid throughout the pipe. The momentum conservation equations 3.3.2 and 3.3.3 are somewhat simplified, and, as explained in chapter 3.4, the pressure correction terms Δp_G and Δp_L serve no useful purpose in a steady-state solution and can be removed. To close the equation system, we obviously need to correlate density, pressure and temperature. Such correlations on general form can be expressed as:

$$\rho_G = \rho_G(p, T) \quad (3.7.1)$$

$$\rho_L = \rho_L(p, T) \quad (3.7.2)$$

When choosing discretization method for the spatial derivatives, it is worth noting that if we knew all boundary conditions in one end of the pipe, we would easily be able to construct a simple recursive, explicit solution scheme. In our example, the gas and liquid mass flows at the inlet are known, and so is the pressure at the outlet. But in a steady-state model, the outlet mass flows are going to be the same as the inlet flows, so we do in fact know both the mass flows and the pressure at the outlet. We do, however, initially not know the pressure at the inlet. It is therefore natural to start the calculations at the outlet, and then compute backwards towards the inlet.

We may use a first order approximation for the spatial derivatives:

$$\frac{\partial(\alpha_G \rho_G v_G^2)}{\partial x} \approx \frac{(\alpha_G \rho_G v_G^2)_i - (\alpha_G \rho_G v_G^2)_{i+1}}{\Delta x} \quad (3.7.3)$$

This leads to the following recursive algorithm:

$$F_{i+1} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}_{i+1} = \begin{bmatrix} (\alpha_G \rho_G v_G)_i - k_{G \text{ in}} \\ (\alpha_L \rho_L v_L)_i - k_{L \text{ in}} \\ (\alpha_G \rho_G v_G^2)_{i+1} - (\alpha_G \rho_G v_G^2)_i + \alpha_{G,i}(p_{i+1} - p_i) - \Delta x F_{G,i+1} \\ (\alpha_L \rho_L v_L^2)_{i+1} - (\alpha_L \rho_L v_L^2)_i + \alpha_{L,i}(p_{i+1} - p_i) - \Delta x F_{L,i+1} \\ \rho_{G,i} - \rho_G(p, T)_i \\ \rho_{L,i} - \rho_L(p, T)_i \\ \alpha_G + \alpha_L - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.7.4)$$

The friction and elevation terms have been written as:

$$\begin{aligned}
 F_{G,i+1} &= -R_{GLi,i+1} + R_{GW,i+1} - \alpha_{G,i+1} \rho_G g \sin \theta_{i+1} \\
 F_{L,i+1} &= R_{GLi,i+1} + R_{LW,i+1} - \alpha_{L,i+1} \rho_L g \sin \theta_{i+1}
 \end{aligned}
 \tag{3.7.5}$$

The variables we seek to determine are:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} \alpha_G \\ \alpha_L \\ v_G \\ v_L \\ \rho_G \\ \rho_L \\ p \end{bmatrix}
 \tag{3.7.6}$$

We start by inserting everything we know at the outlet into vector Y . The pressure is directly inserted as p_{out} . We need to guess a value for α_G , for instance $\alpha_G = 0.5$. That also determines $\alpha_L = 1 - \alpha_G = 0.5$. The densities follow from the fluid properties for the gas and liquid in question according to equations 3.7.1 and 3.7.2.

Next, velocities are determined by equations 3.6.1 and 3.6.2, and all values in Y are thereby known. We index the Y -vector at the outlet Y_{i+1} so that Y_i becomes the nearest upstream grid-point. Using equations 3.4.2, 3.4.5, 3.5.2 - 3.5.12, 3.5.14 - 3.5.16, and 3.7.5, it is straight forward to calculate everything indexed $i+1$ in equation 3.7.4 after we have chosen a discretization length Δx . Y_i can then be determined by Newton-iteration on equation 3.7.4 in the same way as it was described for equation 3.6.9.

The process is repeated throughout the pipe until we reach the inlet end. As starting values in grid point i we may use those from grid point $i+1$, or we may extrapolate from several already calculated grid points.

If we made a poor guess regarding outlet value for α_G , we are still going to approach the correct values some way into the pipe. The inaccurate outlet value will appear as a relatively abrupt change in the fraction. Since values some distance into the pipe are

going to be more accurate, we can use those to extrapolate to the outlet, get better starting values, and do a re-run of the calculations. That procedure can be repeated several times to improve accuracy further.

3.8 Fully transient simulation model

In a fully transient model the time derivatives will obviously no longer be zero, so we need both some sort of spatial discretization as well as discretization in time. Numerical solution methods are discussed in chapter 18, but at this stage, we are only focusing on stratified flow. Without worrying about the complications other situations may bring, we simply conclude that the two-fluid model shown here can be simulated in ways which solve the primary variables (those occurring as time derivatives in the conservation equations, meaning $\alpha_G \rho_G$, $\alpha_L \rho_L$, $\alpha_G \rho_G v_G^2$, and $\alpha_L \rho_L v_L^2$) at every cell for each time-step.

Once the primary variables are determined, we must calculate the secondary variables (those in Y in equation 3.7.6) by some iteration procedure, for instance Newton-iteration. That iteration can be based on equation 3.8.1:

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} \alpha_G \rho_G - (\alpha_G \rho_G) \\ \alpha_L \rho_L - (\alpha_L \rho_L) \\ \alpha_G \rho_G v_G - (\alpha_G \rho_G v_G) \\ \alpha_L \rho_L v_L - (\alpha_L \rho_L v_L) \\ \rho_{G,i} - \rho_G(p, T) \\ \rho_{L,i} - \rho_L(p, T) \\ \alpha_G + \alpha_L - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.8.1)$$

$(\alpha_G \rho_G)$ is the value coming out of the time integration (kept constant during the iteration), while $\alpha_G \rho_G$ are updated for each iteration as α_G and ρ_G converges towards their new values. For the densities, $\rho_G(p, T)$ is the density value calculated by inserting the pressure and (constant) temperature into the gas property equation, while ρ_G is the density as it stands (as it was calculated in the previous iteration). The iterations can be carried out in the same way as described for equation 3.6.11. Initial values can be those from the previous time-step or some extrapolation from several time-steps, or they can be based on values from neighboring grid points or cells.

Equation 3.8.1 is not optimized in any way, and we could for instance insert $\alpha_L = 1 - \alpha_G$ everywhere and eliminate one equation to reduce the work involved in inverting the Jacobi matrix. As explained in chapter 12.8, it is also possible that other iteration methods could converge at a lower cost than Newton-iteration. The purpose here, though, is to show a simple algorithm. Ways of optimizing and generalizing the method are discussed later.

3.9 The drift-flux model

The drift-flux model goes one step on the way to simplifying the full two-fluid model described thus far. Both models are widely used and very similar, but in some ways the drift-flux model is simpler to deal with numerically. In addition, it can be shown that we do not need to include the pressure correction terms Δp_G and Δp_L to maintain hyperbolicity for the drift-flux model.

The drift-flux model combines the two dynamic momentum equations by summarizing them. To maintain closure, the ‘lost’ momentum equation is replaced by an extra algebraic equation.

The mass conservation equations remain equations 3.2.2 – 3.2.4. The dynamic momentum conservation equation is created by neglecting the pressure correction terms and summarizing equations 3.3.2 and 3.3.3:

$$\frac{\partial(\alpha_G \rho_G v_G + \alpha_L \rho_L v_L)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G^2 + \alpha_L \rho_L v_L^2 + p)}{\partial x} = R_{GW} + R_{LW} - (\alpha_G \rho_G + \alpha_L \rho_L) g \sin \theta \quad (3.9.1)$$

Since this equation contains no information about individual forces on each phase, we realize that it cannot fully describe how the velocity difference between the two phases is going to develop. We therefore create an algebraic equation by eliminating $\partial p / \partial x$ between equations 3.3.2 and 3.3.3 (after again having neglected the pressure correction terms). We then take the steady-state, incompressible version of the result, as we also did in equation 3.6.8. That leads to:

$$\frac{R_{GL}}{\alpha_G \alpha_L} - \frac{R_{GW}}{\alpha_G} + \frac{R_{LW}}{\alpha_L} - (\rho_L - \rho_G)g \sin \theta = 0 \quad (3.9.2)$$

This static equation replaces the second dynamic mass conservation equation.

In the two-fluid drift-flux model, one dynamic momentum equation is replaced by an algebraic equation.

By studying the eigenvalues of a linearized version of these system equations it is possible to show that two of them are associated with acoustic waves while the third is associated with a much slower surface wave (though not of the true gravitational sort, since they are not included in the model).

Unlike the model based directly on equations 3.3.2 and 3.3.3, no part of the spatial derivative in equation 3.9.1 has a factor in front of it (in equations 3.3.2 and 3.3.3, the spatial pressure derivatives have α_G and α_L in front of them), so this model is on the same form as the equations we became familiar with in single-phase flow. We can therefore use the Kurganov-Tadmor order 2-scheme (KT2) for spatial discretization directly here, too, just as we could in *Pipe Flow 1*. We will later see that we do not necessarily choose that method, but it offers a simple, explicit solution alternative.

3.10 Ignoring inertia in the momentum equations

We saw in chapter 15.5 in *Pipe Flow 1* that the inertia terms in the momentum equations often can be of little significance in pipelines, it is most often the frictions which dominate. A simplified model based on ignoring inertia obviously cannot describe pressure waves, and it has therefore sometimes been referred to as the *no-pressure-wave model*.

If we neglect everything to do with inertia in equation 3.9.1, we get:

$$\frac{\partial p}{\partial x} = R_{Gw} + R_{LW} - (\alpha_G \rho_G + \alpha_L \rho_L)g \sin \theta \quad (3.10.1)$$

We now have 2 dynamic equations, namely 3.2.2, 3.2.3, in addition to the algebraic equation 3.9.2 and equation 3.10.1, which contains a spatial derivative, though no time derivative.

Simulating these equations is not straight forward, but Patault & Tran (1996) has developed a workable implicit method for doing so. As shown by Viviani (1996) and Masella et al. (1998), the model gives similar results to the full two-fluid model as well as to the drift-flux model for many pipeline situations. Masella et al. (1998) have pointed out, though, that solving these equations seems no more efficient than the (more accurate) drift-flux model, so the no-pressure-wave model appears to be less attractive. Therefore we are not going to go into further details regarding Patault & Tran's (1996) model.

We could in principle go on developing equations to get something similar to what we did for single-phase flow (equation 15.5.11, *Pipe Flow 1*), but we end up with very many terms in the two-phase case. It is unclear whether this procedure would lead to a faster algorithm, and the time it might save is likely to be marginal.

3.11 Incompressible transient model

If we consider both fluids to be incompressible, we avoid having to determine any of the densities, and we can combine the mass conservation equations in the same way as the momentum equations were combined for the drift-flux model. We will see that the pressure term can be eliminated from the main equations, and this model is therefore sometimes called the *pressure-free-model*.

The mass conservation equations 3.2.2 and 3.2.3 can be summarized:

$$\frac{\partial(\rho_G \alpha_G + \rho_L \alpha_L)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G + \alpha_L \rho_L v_L)}{\partial x} = 0 \quad (3.11.1)$$

The momentum equations 3.3.2 and 3.3.3 are written without the pressure correction terms, and then combined in such a way that the pressure is eliminated:

$$\begin{aligned} \frac{1}{\alpha_G} \left[\frac{\partial(\alpha_G \rho_G v_G)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G^2)}{\partial x} \right] + \frac{R_{GLi}}{\alpha_G} - \frac{R_{GW}}{\alpha_G} + \rho_G g \sin \theta \\ = \frac{1}{\alpha_L} \left[\frac{\partial(\alpha_L \rho_L v_L)}{\partial t} + \frac{\partial(\alpha_L \rho_L v_L^2)}{\partial x} \right] - \frac{R_{GLi}}{\alpha_L} - \frac{R_{LW}}{\alpha_L} + \rho_L g \sin \theta \end{aligned} \quad (3.11.2)$$

The terms inside the first bracket can be written as:

$$\left[\frac{\partial(\alpha_G \rho_G)}{\partial t} v_G + \alpha_G \rho_G \frac{\partial v_G}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G)}{\partial x} v_G + \alpha_G \rho_G v_G \frac{\partial v_G}{\partial x} \right] \quad (3.11.3)$$

From the continuity equation, it follows that:

$$\left[\frac{\partial(\alpha_G \rho_G)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G)}{\partial x} \right] v_G = 0 \quad (3.11.4)$$

Notice also that:

$$\frac{\partial(\rho_G v_G)}{\partial x} = \rho_G \frac{\partial(v_G)}{\partial x} + v_G \frac{\partial(\rho_G)}{\partial x} \quad (3.11.5)$$

But since the density is constant, we also have $\partial \rho_G / \partial x = 0$. Therefore:

$$\frac{\partial(\rho_G v_G)}{\partial x} = \rho_G \frac{\partial(v_G)}{\partial x} \quad (3.11.6)$$

This could have been used to eliminate the densities from the two mass conservation equations before we summarized them to become equation 3.11.1. Had we done so, equation 3.11.1 would have taken an alternative form without any of the densities.

Equation 3.11.6 also means:

$$\frac{\partial(\rho_G v_G^2)}{\partial x} = \rho_G v_G \frac{\partial(v_G)}{\partial x} + v_G \frac{\partial(\rho_G v_G)}{\partial x} = 2\rho_G v_G \frac{\partial(v_G)}{\partial x} \quad (3.11.7)$$

If we combine equations 3.11.3, 3.1.4, and 3.11.7, we see that:

$$\frac{1}{\alpha_G} \left[\frac{\partial(\alpha_G \rho_G v_G)}{\partial t} + \frac{\partial(\alpha_G \rho_G v_G^2)}{\partial x} \right] = \left[\frac{\partial(\rho_G v_G)}{\partial t} + \frac{\partial\left(\frac{1}{2}\rho_G v_G^2\right)}{\partial x} \right] \quad (3.11.8)$$

Inserting this into equation 3.11.2, and doing a similar transformation for the terms inside the brackets at the right-hand side, we get:

$$\begin{aligned} \frac{\partial(\rho_L v_L - \rho_G v_G)}{\partial t} + \frac{\partial\left(\frac{1}{2}(\rho_L v_L^2 - \rho_G v_G^2)\right)}{\partial x} \\ = \frac{R_{GL}}{\alpha_L} + \frac{R_{GL}}{\alpha_G} + \frac{R_{LW}}{\alpha_L} - \frac{R_{GW}}{\alpha_G} - (\rho_L - \rho_G)g \sin \theta \end{aligned} \quad (3.11.9)$$

With no compressibility at all in the system, the total volumetric flow at any point in the pipe must constantly equal the total flow injected into the pipe, so:

$$\dot{m}_{G in} + \dot{m}_{L in} = \alpha_G \rho_G v_G + \alpha_L \rho_L v_L \quad (3.11.10)$$

Our unknowns are the velocities v_G and v_L , and the fractions α_G and α_L . Those 4 unknowns can be solved using equations 3.11.1 and 3.11.9, together with the saturation constraint 3.2.4 and the boundary conditions 3.11.10.