

*"Education is the best provision for old age."*  
Aristotle, 350 BC.

# 2 Conservation equations

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**The main equations for transient multi-phase flow with  $N$  phases:**

- ➔ Mass conservation
  - ➔ Momentum conservation
  - ➔ Energy conservation
  - ➔ Mass transfer between phases
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## 2.1 Introduction

In the first book, *Pipe Flow 1*, we learned that 3 conservation equations - mass conservation, momentum conservation, and energy conservation - was sufficient to describe the main conservation principles governing transient single-phase flow. For multi-phase flow, the same three equations apply, but for each phase. Therefore, expect to need 6 equations to describe two-phase flow, 9 to describe three-phase flow, and so on. In practice it does not work quite like that, though, because we take advantage of some simplifications, and some phases may occur in more than one form (in annular flow, there can be both liquid droplets carried by the gas and a liquid film on the pipe wall). We also use additional equations - *closure correlations* - to describe how the phases interact with each other and the pipe wall, as well as to describe the fluid properties.

The simple illustration of multi-phase flow in figure 2.1.1 can for instance symbolize two-phase flow consisting of steam and water. Water can occupy more or less of the available space along the pipe, so we cannot assume the water cross section to be constant. In addition, water may turn into steam or vice versa, so we can no longer assume all water to enter via the control volume's boundaries – some may come from the steam inside the control volume. Although not shown on the figure, the pipe wall may also be perforated and allow fluid to pass through it.

In a more general case, we may have  $N$  different phases rather than only 2. We have seen that those phases can be distributed in several alternative ways – there can be bubbles, droplets, slugs, and various other sorts of fluid distributions. At this stage, though, we simply assume each phase to be continuous, without necessarily taking up the same cross-section everywhere along the pipe.

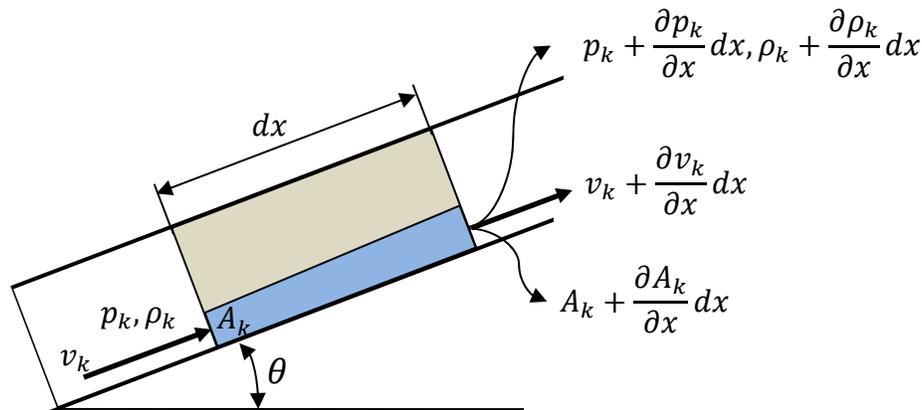


Figure 2.1.1. Compressible multi-phase pipe flow.

## 2.2 Mass conservation

Phase No.  $k$  is assumed to take up cross-sectional area  $A_k$ , while the total pipe cross-section is  $A$ . From that, we define the volume fraction as:

$$\alpha_k \stackrel{\text{def}}{=} \frac{A_k}{A} \quad (2.2.1)$$

Analogy with the single-phase mass conservation equation implies that for each phase  $k$  in a multi-phase flow, the following must generally hold:

$$\begin{aligned}
0 = & \textit{accumulated mass} + \textit{net mass flow into the control volume} + \textit{mass from other phases} + \textit{mass from other sources} \\
0 = & Adx \frac{\partial(\alpha_k \rho_k)}{\partial t} + Adx \frac{\partial}{\partial x} (\alpha_k \rho_k v_k) + \sum_{i=1}^{N, i \neq k} \dot{m}_{ki} + \dot{m}_{kW}
\end{aligned} \tag{2.2.2}$$

The term  $\dot{m}_{ki}$  represents the interface mass flows from each of the other phases into phase  $k$ , and  $\dot{m}_{kW}$  is mass flow into phase  $k$  from other sources such as inflow through perforations in a well. For simplicity, we omit the summation sign, and in effect let  $\dot{m}_{ki}$  stand for the sum of all mass flows into phase  $k$  rather than each component of it.

We re-arrange equation 2.2.2 to:

$$\frac{\partial(\alpha_k \rho_k)}{\partial t} + \frac{\partial}{\partial x} (\alpha_k \rho_k v_k) = \frac{\dot{m}_{ki}}{Adx} + \frac{\dot{m}_{kW}}{Adx} \tag{2.2.3}$$

The continuity equations for each phase are very similar to the one for single-phase flow, but with two extra terms on the right hand side of the equation. They are *volume-specific mass flows*, and we denote them  $\Gamma_k$  [kg/(s·m<sup>3</sup>)]:

$$\boxed{\frac{\partial(\alpha_k \rho_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} = \Gamma_{ki} + \Gamma_{kW}} \tag{2.2.4}$$

The terms  $\Gamma_{ki}$  and  $\Gamma_{kW}$  therefore represent mass transfer per unit volume and time into phase  $k$  from all the other phases and from mass sources.  $\Gamma_{ki}$ , it follows, is in reality a sum of several sources and contains one source from each of the other phases the fluid consists of (although some or all may often be zero). In the special case that no part of the fluid changes phase and nothing flows in through perforations in the pipe wall, we obviously get  $\Gamma_{ki} = 0$  and  $\Gamma_{kW} = 0$ .

Phase change cannot result in altered total mass, so one phase's gain must be another phase's loss. That can be expressed as:

$$\sum_{k=1}^N \Gamma_{ki} = 0 \quad (2.2.5)$$

In order to determine how each fraction appears, it is necessary to know each component's properties. In the simple case that the fluid consists only of water and steam, we have one *component*, H<sub>2</sub>O, and two *phases*. The simulation program must have access to steam and water properties to determine how much condensation or boiling takes place and thereby quantify  $\Gamma_{ki}$ . In our models we generally neglect any chemical time delay involved when gas becomes liquid or vice-versa, and instead simply assume instantaneous equilibrium for the pressures, volumes, and temperatures involved. In chapters 17 and 18 we will discover that it is relatively complicated to model pressure-volume-temperature relationships for real petroleum fluids accurately, and in flow assurance calculations, it is common to use third-party software for that task.

Another useful relation follows from the definition of what a volume fraction is: The sum of all volume fractions must be 1 to fill the pipe's cross section. It is sometimes referred to as the *saturation constraint* and expressed as:

$$\sum_{k=1}^N \alpha_k = 1 \quad (2.2.6)$$

Equations 2.2.4 - 2.2.6 form the basis for mass conservation in multi-phase pipe flow.

### 2.2.1 Comparing single-phase and multi-phase mass conservation

As an example, let us consider single-phase flow as a special case of multi-phase flow by simply setting  $N = 1$ . Equation 2.2.6 implies that  $\alpha_k = \alpha_1 = 1$ . Equation 2.2.5 leads to

$\Gamma_{ki} = 0$ . If nothing flows in through the pipe wall,  $\Gamma_{kW} = 0$ , and equation 2.2.4 simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \quad (2.2.7)$$

As expected, this turns out to be the familiar mass conservation equation for single-phase flow.

### 2.2.2 Mass conservation for well mixed phases

Another interesting special case arises if the different phases are so well mixed that they travel at the same velocity. This can be the situation in a liquid containing a moderate amount of small bubbles, or in a liquid containing droplets of another, immiscible liquid. If the pipe also is un-perforated, then  $\Gamma_{kW} = 0$  for all phases  $k$ .

By combining equations 2.2.4 and 2.2.5 it follows that:

$$\sum_{k=1}^N \frac{\partial(\alpha_k \rho_k)}{\partial t} + \sum_{k=1}^N \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} = \frac{\partial}{\partial t} \sum_{k=1}^N (\alpha_k \rho_k) + \frac{\partial}{\partial x} \sum_{k=1}^N (\alpha_k \rho_k v_k) = 0 \quad (2.2.8)$$

The average density is obviously:

$$\bar{\rho} = \sum_{k=1}^N (\alpha_k \rho_k) \quad (2.2.9)$$

All velocities being identical implies:

$$\bar{\rho} v = \sum_{k=1}^N (\alpha_k \rho_k v_k) \quad (2.2.10)$$

When inserting equation 2.2.9 and 2.2.10 into 2.2.8, we get:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial(\bar{\rho}v)}{\partial x} = 0 \quad (2.2.11)$$

This illustrates that mass conservation for well-mixed multi-phase-flow can be modeled as single-phase by simply using the average density – not a surprising result.

Although not shown here, other equations require us to modify other properties as well if we want to make the model complete. Even a very small amount of air in water, for instance, increases the compressibility and reduces the speed of sound dramatically.

## 2.3 Multi-phase momentum conservation

### 2.3.1 Main equations

Newton's second law applied to one phase can be written as:

$$\begin{aligned} \text{Mass} \quad \cdot \quad \text{acceleration} &= \text{Sum of all forces} \\ A\alpha_k dx \rho_k \quad \cdot \quad \frac{dv_k}{dt} &= \sum F_k \end{aligned} \quad (2.3.1)$$

Ordinary derivatives and partial derivatives relate to each other as:

$$dv_k = \frac{\partial v_k}{\partial t} dt + \frac{\partial v_k}{\partial x} dx \quad (2.3.2)$$

By dividing all terms in equation 2.3.2 with  $dt$  and also defining velocity as  $dx/dt = v$ , we get:

$$\frac{dv_k}{dt} = \left( \frac{\partial v_k}{\partial t} + v \frac{\partial v_k}{\partial x} \right) \quad (2.3.3)$$

We can now insert equation 2.3.3 into 2.3.1:

$$A\alpha_k dx \rho_k \left( \frac{\partial v_k}{\partial t} + v \frac{\partial v_k}{\partial x} \right) = \sum F_k \quad (2.3.4)$$

The derivative of a product can be re-formulated as:

$$\frac{\partial[(\alpha_k \rho_k) \cdot v_k]}{\partial t} = \alpha_k \rho_k \frac{\partial v_k}{\partial t} + v \frac{\partial(\alpha_k \rho_k)}{\partial t} \quad (2.3.5)$$

Hence:

$$\rho_k \alpha_k \frac{\partial v_k}{\partial t} = \frac{\partial(\rho_k \alpha_k v_k)}{\partial t} - v \frac{\partial(\rho_k \alpha_k)}{\partial t} \quad (2.3.6)$$

Similarly:

$$\alpha_k \rho_k v_k \frac{\partial v_k}{\partial x} = \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} - v \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} \quad (2.3.7)$$

Inserting equations 2.3.6 and 2.3.7 into 2.3.4 yields:

$$A dx \left( \frac{\partial(\alpha_k \rho_k v_k)}{\partial t} - v_k \frac{\partial(\alpha_k \rho_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} - v \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} \right) = \sum F_k \quad (2.3.8)$$

By re-arranging the terms, this can be written as:

$$\frac{\partial(\alpha_k \rho_k v_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} - v \left[ \frac{\partial(\rho_k \alpha_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} \right] = \frac{1}{A dx} \sum F_k \quad (2.3.9)$$

The terms in brackets turn out to be the left hand side of the mass conservation equation 2.2.4. Inserting that, we get:

$$\frac{\partial(\alpha_k \rho_k v_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} = v(\Gamma_{ki} + \Gamma_{kW}) + \frac{1}{A} \sum F_k \quad (2.3.10)$$

It is not surprising that the mass transfer terms  $\Gamma_{ki}$  and  $\Gamma_{kW}$  turn up in the momentum conservation equation too. After all, the fluid going from one phase to another,  $\Gamma_{ki}$ , or flowing in through perforations in the pipe,  $\Gamma_{kW}$ , takes its momentum with it.

Thus far we have not discussed exactly which velocity  $v$  symbolizes, but it now appears to represent the velocity for the phases or external sources that transfer mass to phase  $k$ . Since  $\Gamma_{ki}$  is the sum of mass transfer from all other phases,  $v$  must be some sort of average when written in this general form. In practical calculations we would of course insert each phase's velocity. We need to keep in mind also that in our one-dimensional model, only axial momentum is included. This is further elaborated in chapter 3.2.

The different forces acting on the phase must be determined in order to quantify  $\sum F_k$ . As for single-phase flow, there are obviously going to be pressure, gravity, and friction forces. Since phase  $k$  is in contact with other phases, it is no longer only the pipe wall which can exert a friction force on it, and we get additional friction terms.

The next two force contributions may at first seem somewhat unfamiliar: Surface tension and a pressure correction term.

Surface tension is the force that makes it possible for some insects to walk on water. It also makes water 'prefer' to cluster and form drops rather than spread out as individual molecules when it is raining. Surface tension is caused by intermolecular forces trying to attract the molecules towards each other. In the bulk of the fluid, each molecule is pulled equally in all directions by neighboring liquid molecules, resulting in a net force of zero. At the surface, however, the molecules are pulled inwards by other molecules deeper inside the fluid, and they may not be attracted as intensely by the molecules in the neighboring medium (be it vacuum, another fluid or the pipe wall). In that case all molecules at the surface are subject to an inward force of molecular attraction. This results in a higher pressure at the inside. The liquid tries to achieve the lowest surface area possible, similar to a balloon when we pump air into it. Without attempting to dive fully into all details, let us just accept that surface tension can create forces both from other phases to phase  $k$  as well as from the pipe wall to phase  $k$ . A surface interface (a droplet surface, say) may stretch through a control volume boundary, and that can in

principle create some modeling problems, but we will ignore that for now. Also, note that the surface interfacial tension is a physical property with a value depending on the fluids or materials involved. Surface tension between water and vacuum, for instance, is not identical to surface tension between water and a particular hydrocarbon.

The surface tension is generally small compared to many other forces at work, and we will later see that in many cases it can be neglected. It can play a crucial role in some important phenomena, though, for instance the formation of droplets and bubbles or formation of short-wavelength surface waves, and it plays a role in determining the flow regime. This book's models neglect the surface tension forces in the main momentum balances, electing to deal with flow regime transitions in separate criteria. In the general momentum equations developed in this chapter, though, we include surface tension forces mainly to emphasize that they exist.

The various forces acting in phase  $k$  can then be expressed as:

$$\frac{1}{A dx} \sum F_k = F_{k pg} + F_{k g} + R_{ki} + R_{kW} + S_{ki} + S_{kW} \quad (2.3.11)$$

The different forces in the pipe's axial direction on phase  $k$  is as follows:  $F_{k pg}$  is the pressure force due to a pressure gradient along the pipe,  $F_{k g}$  is the gravity force along the pipe due to the pipe's inclination,  $R_{ki}$  is the friction force from other phases (the sum of contributions from each phase),  $R_{kW}$  is the friction force from the wall,  $S_{ki}$  is the force due to surface tension from all other phases (the sum of contributions from each phase), and  $S_{kW}$  is the surface tension force from the wall. Note that since  $\sum F_k$  has the term  $1/A dx$  in front of it, all the force terms on the right-hand side of equation 2.3.11 are on the volume-specific form, meaning they are forces pr. unit volume of pipe [ $N/m^3$ ].

Finding good approximations for each of the forces to be inserted into equation 2.3.11 is not easy. A correlation for the pressure force  $F_{k pg}$ , for instance, can in principle be found by looking at figure 2.1.1 and expressing the forces at each end of the studied element. That works fine as long as the phases are separated in layers in the way the figure indicates, but would obviously not be as straight-forward if one or several phases appeared as bubbles or droplets. Ignoring that problem for now, we simply calculate the volume-specific net force in the axial direction due to pressure on the surfaces on phase  $k$  as:

$$F_{k\,pg} = \frac{A\alpha_k p_k + F_{k\,int\,x} - A\left(\alpha_k + \frac{\partial\alpha_k}{\partial x} dx\right)\left(p_k + \frac{\partial p_k}{\partial x} dx\right)}{A dx} \quad (2.3.12)$$

$F_{k\,int\,x}$  is the internal axial force on phase  $k$  due to the pressure on the surface inside the control volume. That force depends both on how the *pressure on* and *the shape of* the phase vary from inlet to outlet of the control volume, and they are both unknown. As an example, consider a situation where the pressure on the outside of phase  $k$  and the fraction of phase  $k$  vary linearly from the control volume's inlet to outlet. In that case, the average pressure will be  $p_k + 1/2 \partial p_k / \partial x \cdot dx$ , and the effective area on which the axial force works is  $A \partial \alpha_k / \partial x \cdot dx$ . Inserting that into equation 2.3.12, we get:

$$F_{k\,pg} = -\alpha_k \frac{\partial p_k}{\partial x} - \frac{1}{2} \frac{\partial \alpha_k}{\partial x} \frac{\partial p_k}{\partial x} dx \quad (2.3.13)$$

If we assume other pressure distributions, we will see that it only alters the factor in front of the last term in equation 2.3.13. If we ignore higher order terms, any axial pressure distribution inside the control element would lead to the same result, namely:

$$F_{k\,pg} = -\alpha_k \frac{\partial p_k}{\partial x} \quad (2.3.14)$$

The volume-specific gravity-force is very similar to how it was for single-phase flow:

$$F_{k\,g} = \frac{-A\alpha_k dx \rho_k g \sin \theta}{A dx} \quad (2.3.15)$$

This can be expressed as:

$$F_{k\,g} = -\alpha_k \rho_k g \sin \theta \quad (2.3.16)$$

The other forces are less meaningful to express in a general way and have to be considered after the flow regime has been determined. If we simply combine equations 2.3.10, 2.3.11, 2.3.14 and 2.3.16, we get:

$$\begin{aligned}
 & \frac{\partial(\alpha_k \rho_k v_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} \\
 & = -\alpha_k \frac{\partial p_k}{\partial x} + (R_{ki} + R_{kW}) + (S_{ki} + S_{kW}) \\
 & + (v_{ki} \Gamma_{ki} + v_{kW} \Gamma_{kW}) - \alpha_k \rho_k g \sin \theta
 \end{aligned} \tag{2.3.17}$$

### 2.3.2 Pressure differences between phases due to elevation differences

Equation 2.3.17 on the form it stands here uses separate pressures for all phases. It seems reasonable to assume that in our one-dimensional model we could adopt the same pressure in all phases, and therefore set  $p_k = p$  for all  $k$ . It turns out that this is not always a good idea, and we sometimes introduce a *pressure correction term* to describe this pressure difference between phases. The term itself is discussed in greater detail in chapters 3.4 (for two-phase flow) and 15.3 (for three-phase flow).

To conceptualize why, consider the case of stratified flow shown on figure 2.1.1, with liquid at the bottom and gas on top. The liquid will experience a slightly higher pressure than the gas since it is at a lower elevation (it has more fluid on top of it). This elevation difference is often a very small one compared to the axial elevation variations resulting from the pipe not being horizontal, and it is tempting to think - as did developers of early multi-phase models - it can be neglected.

The problem is that surface waves on the liquid, like those on a lake, can only be modeled adequately if we take into account the gravitational pressure differences under a wave top compared to a wave bottom. If we neglect the pressure correction term, we cannot expect the model to reproduce surface waves accurately, and they are crucial in the mechanisms at work when stratified flow switches to slug flow.

The biggest problem with neglecting the pressure correction term, though, is that it can cause our model to lose its hyperbolicity, and that can in turn lead to numerical

problems. Interestingly, choosing spatial discretization grids with  $\Delta x$  exceeding the surface wavelength, something which also makes the model lose its ability to reproduce surface waves, does not lead to similar numerical problems.

As we will see later, there are also ways of simplifying these general momentum equations which avoid the loss-of-hyperbolicity problem. It is even possible to regard the pressure correction term as a minor correction introduced for the sole purpose of making the equations hyperbolic whether or not the term corresponds to physical reality (Evje & Flatten, 2005). The subject of whether multi-phase flow equations are hyperbolic, as they should be, has been investigated by a numerous authors, for instance Bonizzi & Issa (2003), Bouchut et al. (2000), and Cortes et al. (1998). As a general rule of thumb, the model tends to end up with the required hyperbolicity if those phenomena it describes have been modeled adequately. Otherwise the model tends to become *ill posed*.

A model referred to as the *drift-flux model* does also indirectly incorporate some dynamics for the surface area, although it does so in a less dynamic way (one of the momentum equations is replaced by a static equation), and that turns out to make the model robust even if the pressure correction terms are omitted. We can therefore not reverse the argument and claim the model necessarily loses its hyperbolicity if any details are approximated in a physically incorrect way. Any model rests on various simplifications, making it difficult to utilize the rule of thumb in practice - we typically do not know how well each phenomenon needs to be included to avoid problems. Other techniques - eigenvalue analysis on a linearized version of the system, say - are required to investigate a model more thoroughly, though as a mental reference, this rule of thumb can be useful.

**As a general rule of thumb the conservation equations tend to end up with the required hyperbolicity when the phenomena described have been modeled accurately.**

Some implicit numerical methods are robust enough to churn along even if the hyperbolicity is lost, and it took some time to recognize the problem initially. In fact this is one of the dangers of implicit integration algorithms: They may have so much numerical damping built-in they are stable even when the physical systems they simulate are not, and real, physical instabilities as well as ill-posed model formulation

can remain undetected. As a general rule, though, we cannot set all  $p_k$  equal unless we know the numerical consequences of doing so.

Assuming the pressure to be identical in all phases can lead to numerical problems in some models.

If we choose focusing solely on the model's steady-state solution, on the other hand, we neglect all time derivatives, and such a model obviously cannot replicate surface waves (or any other sorts of transient phenomena for that matter), and including the pressure correction terms serves no purpose.

Notice that introducing a pressure correction term which expresses radial pressure differences does actually make our model quasi 2D, since we take into account only one (not all) of the multidimensional phenomena.

### 2.3.3 Summarizing the forces between phases

Equation 2.2.5 expresses that the sum of mass flows from phase to phase have to be zero: Mass added to one phase must disappear from another. A similar correlation must apply to forces: Any force acting on phase  $k$  from other phases must have an opposite counterforce on those other phases. The sum of all forces between different phases must therefore be zero. In our model, there are only three such forces: The momentum exchange, the surface tension, and the friction forces. Summarizing for all  $N$  phases, we get:

$$\sum_{k=1}^N R_{ki} + S_{ki} + v_k \Gamma_{ki} = 0 \quad (2.3.17)$$

This momentum conservation equation must therefore be satisfied in addition to equation and 2.3.16.

### 2.3.4 Comparing single- and multi-phase momentum conservation

For single-phase flow,  $\alpha_k = 1$ , and no mass transfer or forces can exist between phases. That means  $\Gamma_{ki} = 0$ ,  $R_{ki} = 0$  and  $S_{ki} = 0$ . If nothing flows in through the pipe wall, we can also set  $\Gamma_{kW} = 0$ . Neglecting surface tension forces between the fluid and the pipe wall means  $S_{kW} = 0$ . The friction force per unit volume of fluid can be expressed as:

$$R_{kW} = -\frac{A\rho \frac{f dx}{2d} v|v|}{A dx} \quad (2.3.18)$$

We can drop the  $k$ -index when we only have one phase. Inserting all this into equation 2.3.17, we get:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2)}{\partial x} = -\frac{\partial p}{\partial x} - \frac{f\rho}{2d} v|v| - \rho_k g \sin \theta \quad (2.3.19)$$

As expected, this result is identical to the momentum equation previously found for single-phase flow (equation 6.2.12 in *Pipe Flow 1*).

## 2.4 Energy conservation

As for single-phase flow, each phase's accumulated energy must equal the net sum of what that phase brings in minus what it lets out, plus any heat and work added from the outside. In addition, it is possible for the phase to receive heat and work, and also material (containing energy) from other phases. With an eye to equation 6.3.12 in *Pipe Flow 1*, we can directly write:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \alpha_k \rho_k \left( u_k + \frac{v_k^2}{2} + g z_k \right) \right] \\ &= -\frac{\partial}{\partial x} \left[ \alpha_k \rho_k v_k \left( h_k + \frac{v_k^2}{2} + g z_k \right) \right] + q_{ki} + q_{kW} + w_{ki} + w_{kW} \quad (2.4.1) \\ &+ \Gamma_{ki} h_{ki} + \Gamma_{kW} h_{kW} \end{aligned}$$

$q_{ki}$  is specific heat from other phases to phase  $k$ ,  $q_{kW}$  is specific heat from the wall to phase  $k$ ,  $w_{ki}$  is specific work from other phases on phase  $k$ ,  $w_{kW}$  is specific work from the outside on phase  $k$ ,  $\Gamma_{ki}$  is specific mass flow from other phases into phase  $k$ ,  $\Gamma_{kW}$  is

specific mass flow from other sources into phase  $k$  (such as via perforations in the pipe), and  $h_{ki}$  and  $h_{kW}$  are the relevant specific enthalpies.

This can be brought over to a slightly more compact form by defining the fluid's internal energy per unit volume as:

$$E \stackrel{\text{def}}{=} \rho \left( u + \frac{v^2}{2} + gz \right) \quad (2.4.2)$$

and specific enthalpy:

$$h \stackrel{\text{def}}{=} u + \frac{p}{\rho} \quad (2.4.3)$$

We then get:

$$\frac{\partial}{\partial t} (\alpha_k E_k) = - \frac{\partial}{\partial x} [\alpha_k v_k (E_k + p_k)] + q_{ki} + q_{kW} + w_{ki} + w_{kW} + \Gamma_{ki} h_{ki} + \Gamma_{kW} h_{kW} \quad (2.4.4)$$

The sum of all specific heat from other phases to phase  $k$  must obviously summarize to zero, since heat transferred from other phases by definition must be heat lost for those other phases:

$$\sum_{k=1}^N q_{ki} = 0 \quad (2.4.5)$$

The sum of all specific heat being received from the wall must also summarize to the total amount of heat flowing through the wall from the environment:

$$\sum_{k=1}^N q_{kW} = q \quad (2.4.6)$$

Similar for specific work from phase to phase:

$$\sum_{k=1}^N w_{ki} = 0 \quad (2.4.7)$$

The sum of all specific work received from the outside by each phase must equal the total work added from the outside:

$$\sum_{k=1}^N w_{kW} = w \quad (2.4.8)$$

The sum of specific enthalpy transferred in the mass flow from phase to phase must equal zero:

$$\sum_{k=1}^N \Gamma_{ki} h_{ki} = 0 \quad (2.4.9)$$

### 2.4.1 Comparing single-phase and multi-phase energy conservation

Just as we did for the mass and momentum conservation equations, we may compare the general multi-phase energy equation 2.4.4 with the one for single-phase flow. Introducing the same modifications as explained in chapter 2.3.1, in addition to setting  $q_{ki} = 0$ , we get:

$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial x} [v(E + p)] + q + w \quad (2.4.10)$$

As expected, this is the same as equation 6.3.12 in *Pipe Flow 1* when the definition of  $E$ , equation 2.4.2, and  $h$ , equation 2.4.3, is inserted.

## 2.5 Mass transfer between phases with equal pressures

We have demonstrated that mass transfer between phases takes part in all the three conservation equations. The terms appear with different notations in various literature, but we have chosen to use  $\Gamma_{ki}$ , which is mass transfer per unit volume and time into phase  $k$  from all other phases combined. The phase change causing this mass transfer needs to be quantified. How can that be done?

We begin by assuming phase transfer takes place instantly according to the fluid's properties as the pressure or temperature changes, neglecting any delays in the phase change itself. The *flash calculation* - the chemical calculation to determine the new equilibrium after the pressure and/or temperature has changed determines liquid density, gas density, gas and liquid fractions, the resulting composition in each phase, and various other properties. For the purpose of the flash calculations (but not necessarily for the flow calculations) we usually neglect the pressure difference between phases since it has minimal effect on the chemical equilibrium.

For the volume-specific mass of phase  $k$ ,  $m_k$ , mass increase can come from other phases or from inflowing fluid:

$$\frac{dm_k}{dt} = \dot{m}_{ki} + \dot{m}_{kW} \quad (2.5.1)$$

We define the mass fraction for phase  $k$  as:

$$\alpha_{m k} = \frac{m_k}{m_{total}} = \frac{m_k}{\sum_{i=1}^N m_i} = \frac{m_k}{\sum_{i=1}^N \alpha_i \rho_i} \quad (2.5.2)$$

If nothing flows into the control volume through the pipe's wall, we can set  $\dot{m}_{kW} = 0$ , and we get:

$$\Gamma_{ki} = \frac{dm_k}{dt} \quad (2.5.3)$$

Inserting equation 2.5.2 into 2.5.3:

$$\Gamma_{ki} = \frac{d\alpha_{mk}}{dt} \sum_{i=1}^N \alpha_i \rho_i \quad (2.5.4)$$

If the mass fraction is a function of pressure and temperature:

$$\alpha_{mk} = \alpha_{mk}(p, T) \quad (2.5.5)$$

Note that single-component fluids (for instance pure water) do not fit well into equation 2.5.5, since it boils at a particular temperature for a given pressure (approximately  $100^\circ\text{C}$  at  $1 \text{ atmosphere} \approx 10^5 \text{ Pa}$  in case of water), regardless of the mass fraction of gas and liquid, meaning pressure and temperature alone cannot determine fractions. The equation developed below is therefore not applicable to single-component fluids. One may overcome this by expressing the fluid properties as function of  $p$  and enthalpy  $h$  rather than  $p$  and  $T$ , since  $h$  changes as more and more fluid boils or condenses even for a single-component fluid. That makes the theory more general. When simulating single-component  $\text{CO}_2$  pipe flow, a subject of increasing interest to technologies for limiting the release of greenhouse gases, this simple modification provides a good alternative.

In the following, we will stick to equation 2.5.5 and set:

$$d\alpha_{mk} = \left( \frac{\partial \alpha_{mk}}{\partial p} \right)_T dp + \left( \frac{\partial \alpha_{mk}}{\partial T} \right)_p dT \quad (2.5.6)$$

And hence:

$$\frac{d\alpha_{mk}}{dt} = \left( \frac{\partial \alpha_{mk}}{\partial p} \right)_T \frac{dp}{dt} + \left( \frac{\partial \alpha_{mk}}{\partial T} \right)_p \frac{dT}{dt} \quad (2.5.7)$$

Similarly we can easily show:

$$\frac{dp}{dt} = \left( \frac{\partial p}{\partial t} \right)_x + \left( \frac{\partial p}{\partial x} \right)_t \frac{dx}{dt} \quad (2.5.8)$$

And:

$$\frac{dT}{dt} = \left(\frac{\partial T}{\partial t}\right)_x + \left(\frac{\partial T}{\partial x}\right)_t \frac{dx}{dt} \quad (2.5.9)$$

The term  $dx/dt$  is the definition of velocity, and it applies to phase  $k$ :

$$v_k = \frac{dx}{dt} \quad (2.5.10)$$

By inserting equations 2.5.7-2.5.10 into 2.5.4, we finally get:

$$\Gamma_{ki} = \left\{ \left(\frac{\partial \alpha_{mk}}{\partial p}\right)_T \left[ \left(\frac{\partial p}{\partial t}\right)_x + \left(\frac{\partial p}{\partial x}\right)_t v_k \right] + \left(\frac{\partial \alpha_{mk}}{\partial T}\right)_p \left[ \left(\frac{\partial T}{\partial t}\right)_x + \left(\frac{\partial T}{\partial x}\right)_t v_k \right] \right\} \sum_{i=1}^N \alpha_i \rho_i \quad (2.5.11)$$

The two partial derivatives in front of each of the square brackets in equation 2.5.11 are fluid properties, so they can be looked up from tables or calculated indirectly from such well-known correlations as the *Peng-Robinson* or the *Redlich-Kwong* equations of state, in combination with *mixing rules*. They are then fed into equation 2.5.11 as numbers, so it is only the other partial derivatives which take part in the discretization to solve the equation.

When calculating the mass transfer for each phase this way we should end up with something which satisfies equation 2.2.5. If we don't, it means we have inaccuracies in our calculations, and it is best to modify the result to make it fit equation 2.2.5 perfectly so that mass conservation is not violated.

## 2.6 Comments on the conservation equations

### 2.6.1 Averaging

When developing the equations in this chapter, we have presumed all phases to be continuous. For many of the flow regimes described in *chapter 1*, that assumption does clearly not hold. Droplets and bubbles, for instance, can be very small, and when one of

them passes a control volume boundary, we have a different situation compared to when they are completely inside the boundary. The flow can therefore contain discontinuities, and strictly speaking, the derivatives may at times not be defined. This problem is usually solved by doing some sort of averaging, and numerous papers and books deal with this subject, including Ishii (1975), Yadigaroglu & Lahey (1976), Mathers et al. (1978), Nigmatulin (1979), Drew (1983), Lahey & Drew (1988), Daniels et al. (2003), Ishii & Hibiki (2006), Prosperetti & Tryggvason (2007), and Jacobsen (2008).

The most common forms of averaging are:

1. Spatial (volume or area) averaging, with no averaging in time.
2. Time averaging, with no spatial averaging.
3. Ensemble averaging, which is a statistical way to average. It can be regarded as a measure related to the repeatability of experiments (Jacobsen, 2008).
4. A combination of several of the above, such as ensemble/space averaging or time/space averaging.

The averaging process acts as a filter removing information occurring below certain length and time scales, and it smoothens out discontinuities. The averaged equations will only be able to resolve flow features down to the limits defined by the averaging process. This is not a serious problem when simulating long pipelines or wellbores, but with an exception for intermittent flow, since a slug can stretch over several grid-points

**Since multiphase flow contains discontinuous phenomena, in reality the equations used are based on some sort of averaging in time and/or space. Of the models shown here, averaging issues require attention solely for intermittent flow.**

or cells. Rather than going into details on all the various averaging literature, for our purpose it is sufficient to point out that the equations developed in this chapter are valid for most situations. Later we will discuss modifications for slug or churn flow.

### 2.6.2 Closure relationships

So far we have not brought in any fluid-specific properties, such as how viscosity, density, surface tension, or specific enthalpy varies with pressure and temperature. In that sense both the mass conservation and the momentum equations are general in the

form they are shown here and they are valid for any fluids, but at the cost of being incomplete. We realize that other correlations must also be added, to describe friction or heat, for instance. Those extra correlations are often referred to as *closure relationships*, since they are required to close the equation set so the number of unknowns equals the number of equations.

We also need to establish flow regime criteria and find ways to solve the equations. As we will discover in following chapters, both tasks offer many interesting challenges. We end up with very different models depending on how we deal with those challenges, even though all models rest on the conservation principles shown here in chapter 2.